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Proof by Mathematical Induction

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***Abstract* – On this paper we deal with the concept of proof by mathematical induction. Mathematical induction is one of the most powerful tools for proving statements in discrete mathematics. Notwithstanding, it is studied only by students that learn the highest level of mathematics at school, even then it is taught in the most shallow way. Like in many subjects of mathematics, the student is being taught how to use the technique to prove simple propositions of certain form, without any theoretical basis to understand the validity of it. We feel that one way to achieve a better understanding of the induction can be obtained by exposing the students to a greater variety of propositions. For that matter we end this paper with some examples of varies statements proved using mathematical induction.**

Furthermore, it is rather odd that the concept of complete mathematical induction, which will be discussed further in this paper, and which is even a more powerful tool, is not taught at school at all.

***Keywords* – Induction, Misconceptions, Proof, Complete Induction, Paradox, Mathematical Induction.**

I. Introduction

The method of proof by induction is taught in high schools only superficially, only with respect to very specific kinds of statements. The students learn how to prove certain kinds of identities and inequalities, as well as properties of division by natural numbers. No focus is placed on the correctness of the induction principle; there is no variety in the kind of statements treated; and there is no room for originality.

After extensive observations of novice math students, we can say that although most students successfully employ the method of proof by induction to prove statements of the kind they are accustomed to, they do not understand the correctness of the proof, see [1]. Most students learn how to use the method mechanically and since they lack an understanding of the correctness of the method, they fail in solving problems that are of a different style.

In general, no emphasis is placed in schools on the concept of *proof*. Even students who study mathematics at high level encounter the general concept of proof only when dealing with the topics of geometry (with respect to overlapping triangles), when dealing with trigonometric identities and proof by induction. In the framework of these three topics, students learn how to use proofs in a systematic manner, devoid of any originality. It is therefore not surprising that when asked, at a more advanced stage, to prove something, some students are simply clueless. It is no wonder that many engineering students, and even math students, stand gaping when the word "Prove" appears at the beginning of a test question.

Often, when students are asked to prove that A causes B, they prove the opposite. Furthermore, they are not even aware that the important tool - indirect proof - even exists. The concept of proof in general is a very broad and interesting concept. In this paper, however, we will focus only on the concept of *proof by induction*.

Many articles deal with the differences between mathematics and other sciences and one of the links between them, seemingly, has to do with the method of proof by induction. Physical principles are proved by repeating experiments dealing with a specific phenomenon. From the point of view of the physical scientist, any phenomenon that can be obtained repeatedly can serve as proof of a law or rule, which can even be formulated as a mathematical equation. Such proof is called an *inductive proof*. Many students think that this concept is valid with respect to mathematics as well, and that it is applied in the induction principle. Shmuel Avital[2] calls this approach *erroneous induction*, while Adam Kenisberger [9] calls it *empirical induction*. Had this approach been accepted by mathematicians, then Goldbach's assumption, which claims that "every natural, even number $n > 2$, can be represented as the sum of two prime numbers", would have long ago become a theorem, since it has already been proved to be true for all $n \le 2 \cdot 10^{17}$. Furthermore, Ferma's theorem would not, most likely, require its very complex proof, which extends over some 250 pages, and which states that "for any natural $n > 3$, no three natural numbers x, y, z exist, such that $x^n + y^n = z^n$".

II. Proof by Induction

The induction principle can be described by means of the following story:

It is well known in the country of Mathland that if a person takes an evening stroll one day, then this same person will take an evening stroll the next day as well. Tom lives in Mathland. Can we say, following the above, that Tom takes an evening stroll every day? The answer is, of course, no. We are missing one additional piece of information. We must know whether Tom took an evening stroll on any particular day before. If he did, then from that day on, he will indeed take a stroll every day. This is exactly the principle of induction.

When examining the extent of students' understanding, who supposedly already knows how to use the principle of induction, the following problems are soon encountered:

A. First Problem: Misunderstanding of the Deductive Proof.

After proving the basis of the induction, we must then prove that the correctness of the statement for $n+1$

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follows from its correctness for n. At this point, students sometimes ask: "But how do we know that the statement is true for n?" or "But proving the statement for n is exactly the objective we aspired to achieve, isn't it?". Many students think that at this stage they must prove the correctness of the statement for the case of $n+1$, and do not understand that this is not so. It seems that most of the students do not know that at this stage only the deduction must be proved.

B. Second Problem: Incomplete Comprehension of the Meaning of the Basis of Induction.

The symptoms of this misunderstanding are:

- Testing the correctness of the statement for an unnecessary number of cases. Students will sometimes test the correctness of the statement for $n=1$ and for $n=2$, whereas it would have sufficed to test it only for $n=1$. There are many possible reasons why a student would do so. The student's high school teacher might have done so, or the student might have a recollection of his teacher doing so. Indeed, there are cases that do require testing the correctness of the statement for both $n=1$ and $n=2$ (due to the size of increment, or if the proof is of the kind illustrated hereafter in Problem No. 1). In any case, any student who tests the correctness of a statement for two consecutive cases when there is no need to do so, probably lacks an understanding of the induction concept.
- Not knowing which basis of induction to choose. Students are accustomed to proving statements for any natural n. When they are asked to prove a statement, say for any natural $n \geq 3$, or for any even number, they have difficulties deciding for themselves what the basis of induction should be. After being told by the teacher what the basis of induction is for this specific case, the student will "input" the case into a kind of mental table that correlates each case with its corresponding basis of induction.

C. Third Problem: Misunderstanding of the Increment

When a statement is to be proven for any natural n, the increment is 1. For other cases, however, other increments can be appropriate. If, for instance, we wish to use induction to prove a statement for any even, or odd, natural number, the increment will be 2. After a while, students learn, for example, that when using proof by induction for any even n, theymust use an incrementof2. However, upon encountering a problem whereby they must prove a statement for every odd n, students use an increment of 1. And when faced with a problem that requires the proof of a statement for all n that are divided by 3, students simply give up.

A student who understands the underlying logic of the induction principle can apply this principle more freely. For instance:

- In certain cases it is better to use increments that are greater than 1, say an increment of size k. In this case, the correctness of the statement must be tested for $n=1,\ldots,k$.
- Induction can be used for any set that is equivalent to the set of natural numbers. For example, the set of negative integers. In this case, the basis of induction will be $n=-1$ and it must be proved that the correctness of the statement for $n-1$ follows from the statement's correctness for n.
- The induction principle can be also used on two variables. Suppose we want to prove that the statement $P(m,n)$ is true for any natural $m$and n. In this case, we start by proving the basis of induction for one of the variables, say n. To do so, the correctness of $P(m,1)$ must be demonstrated for any natural m. If possible, this can be shown using induction with respect to m. Finally, it must be proven that the correctness of the statement $P(m,n+1)$ follows from the correctness of $P(m,n)$, for any m and n.
- There are cases in which induction can be used on a variable that is hidden in the problem. Students are usually accustomed to applying the principle of induction on variables that appears in the problem, but sometimes a greater degree of originality is called for. At times, a new variable must be defined, and the principle of induction applied to it. This kind oforiginality slightly resembles that which must be exhibited when one is required to useauxiliary construction as the initial stage in a geometric proof.

III. Paradoxes in Induction

One way to better sense where the gaps in the students' understanding of the induction principle is to present erroneous proofs by induction of erroneous statements and ask the students to identify the mistakes in the proofs. We do not suggest that this exercise be used with students who are just beginning their journey through the topic; it might cause them to lose faith in the induction principle, faith that is already rather shaky.

We will present two famous "paradoxes", each with a different problematic point.

A. PARADOXNO. 1.

Statement: For any given group of people, the height of all of the people in the group is uniform.

Proof: Proof by induction on n, where n is the number of people in the group.

For $n=1$, the statement is obviously true, since in any group that contains only one person, the height of the people in the group is uniform (and equals the height of the said person).

We assume the statement is true for any natural n and prove it for $n+1$.

Let A be a group of $n+1$ people. We will prove that the height of the people in group A is uniform. Let us remove a certain person, x, from group A. As a result, we have now a group containing n people, which we denote by B. According to the induction assumption, the height of the people in group B is uniform. Let us assume that this height is equal to b centimeters. Now, let us take group A again and remove a different person from it, denoted by y. The group obtained will be calledgroup C. Sincegroup C contains n people, the height of the people in this group is also uniform. Let us say that this height is equal to c centimeters. Since the height of the people in group A, excluding persons x and y, is both b and c centimeters, it follows that $b = c$. Furthermore, this is also the height of persons x and y since each of them belongs either to group B or to group C. In other words, the height of the people in group A is uniform.■

The problem with this proof lies in the deductive proof. The process whereby the correctness of the statement for $n+1$ follows from the correctness of the statement for any natural n is not valid for $n=1$. In this case, $n+1=2$. If we remove two different people, x and y, from a group containing 2 people, the remaining group will be empty. However, the uniformity of the heights of people x and y was based on the group of people who are part of both groups, B and C.

Students who encounter this erroneous proof usually fail to identify the error in the proof and it is very interesting to listen to their attempts to explain the source of the error. A significant part of the students claim that the problem lies with the induction assumption and their claim usually sounds something like this: "The problem is with the assumption that in a group of n people, the height is uniform. We know that this is not true".

B. PARADOX NO. 2.

Statement: If a and b are natural numbers, then $a = b$.

Proof: We will prove the statement for n, where $n = max\{a,b\}$.

If $n=1$, then $a=b=1$, and we are done.

We will assume the correctness of the statement for any natural n and prove it for $n+1$.

Let a and b be natural numbers, for which $max\{a,b\} = n+1$. We now prove that $a=b$.

Since $max\{a-1,b-1\} = n$, it therefore follows from the induction assumption that $a-1=b-1$; thus $a=b$.■

The problem with this proof is that the fact that a and b are natural numbers does not lead to the conclusion that $a-1$ and $b-1$ are natural numbers as well.

Thus, the induction assumption may not be used for $max\{a-1,b-1\} = n$.

IV. Complete Induction

It is interesting to note that the principle of induction is taught at school, but the principle of complete induction, which is a more powerful tool for proving statements, is not. Many statements can be proved using complete induction but cannot be proved using regular induction.

The principle of complete induction is equivalent to the minimum value principle (see Cooperman [3]), which states that in every non-empty set of natural numbers there exist a number that is the smallest. The following statement, which is slightly humorous, but nonetheless true, can also be proved using this principle.

Statement: All natural numbers are special.

Proof: Let A be the set of all non-special natural numbers. We will prove that A is an empty set. Assume that A is not empty, then according to the minimum value principle, there exist $a = min\,A$. In other words, a is the smallest non-special number, and this is a very special property - contradiction.■

The principle of the complete induction states that if the correctness of a certain statement with respect to any natural n follows from the correctness of the statement for any natural k that is smaller than n, then the statement is true for all natural numbers. We will demonstrate this principle by means of an example:

Statement: For any natural number, $n \ge 2$, there exists a prime number, p, that is a divisor of n.

Proof: Assume the statement is true for any natural k, where $n > k \ge 2$. We will prove the statement for n. If n is prime, we are done. Otherwise, there exist $2 \le a,b < n$, such that $n = a \cdot b$. According to the induction assumption, there exists a prime number, p, which is a divisor of a. Hence, p is also a divisor of n.■

V. Selected Problems to be Proved by Induction

This section presents a variety of examples that can be proved by induction. An additional example with its full solution can also be found in H. David's paper [4].

Problem No. 1. Given the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, prove that $A^n = 3^{n-1} \cdot A$ for any natural n.

Proof: Proof by induction on n.

The statement is true for $n = 1,2$ since $A = 3^0 \cdot A$ and $A^2 = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = 3 \cdot A$.

We will assume the correctness of the statement for any natural $n \ge 2$ and will prove the statement for $n+1$.

$$A^{n+1} = A^n \cdot A \overset{assumption}{=} 3^{n-1} \cdot A \cdot A \overset{n=2}{=} 3^{n-1} \cdot 3 \cdot A = 3^n A \blacksquare$$

Note: The basis of induction of the previous problem required that we test the case of $n=2$ as well, since it was to be used in order to prove the deduction.

Problem No. 2. Based on the fact that the sum of the interior angles of a triangle is 180°, prove that the sum of the interior angles of any convex polygon with $n \ge 3$ vertices is $180 \cdot (n-2)°$.

Proof: Proof by induction on n.

For $n=3$, the polygon is a triangle and the statement is true.

We will assume the correctness of the statement for any natural $n \ge 3$, and we will prove the statement for $n+1$. Take a polygon with $n+1$ vertices and draw a diagonal that connects two non-adjacent vertices that are separated by only one other vertex. This diagonal divides the polygon into two polygons: one that is a triangle and another that is a polygon with n vertices. The sum of the interior angles of the polygon with $n+1$ vertices is equal to the sum of the angles of the two sub-polygons. This sum is equal to 180° for the triangle, plus $180\cdot(n-2)^{\circ}$ for the polygon with n vertices, according to the induction assumption. In other words, a total of $180\cdot(n-1)^{\circ}$, as required.■

Note No. 1: The complete induction proof presented for the previous problem could rely on any diagonal, in any n-vertex convex polygon, that divides the polygon into two sub-polygons, with k vertices and with m vertices, where $k,m<n$ and $k+m=n+2$. According to the induction assumption about k and m, we would find that the sum of the interior angles of a polygon with n vertices is

$$180\cdot(k-2)+180\cdot(m-2)=180\cdot(m+k-4)=180\cdot(n-2)$$

degrees, as required.

Note No. 2: The above statement is true also for non-convex polygons, but the proof is slightly more complex.

Before proceeding to the next problem, we present three set theory notations.

Notation: For a finite set A, let $|A|$ denote the number of terms in the set.

Notation: For any set A, let $P(A)$ denote the set of all partial sets of A.

Notation: Let ϕ denote the empty set.

Problem No. 3. Prove that the number of partial sets of a set with n terms is 2^n.

Proof: Proof by induction on n.

The statement is true for $n=0$, because a set with 0 terms is necessarily the empty set and the empty set has only one partial set, which is the empty set itself. In other words, $|P(\phi)|=1=2^0$, as required.

We will assume the correctness of the statement for any integer $n\geq 0$ and we will prove the statement for $n+1$. Let A be a set with $n+1$ terms.

Since the number of terms in A is at least 1, we can take any term in A and denote it x. Let $P_x(A)$ denote the set of all partial sets of A that contain x and let $P_x^*(A)$ denote the set of all partial sets of A that do not contain x. It is clear that $|P(A)|=|P_x(A)|+|P_x^*(A)|$.

Note that $P_x^*(A)$ is actually the set, $P(A-\{x\})$, of all partial sets of a set that contains n terms. Therefore, according to the induction assumption, $|P_x^*(A)|=2^n$.

Furthermore, if we add the term x to each of the sets in $P_x^*(A)$, we will obtain the set $P_x(A)$. In other words, $|P_x(A)|=|P_x^*(A)|=2^n$.

Hence, $|P(A)|=|P_x(A)|+|P_x^*(A)|=2\cdot 2^n=2^{n+1}$.■

Note: In the previous exercise we proved the statement for all non-negative integers, since the basis of induction was for $n=0$.

A Fibonacci sequence is a sequence of natural numbers, which has a special significance in mathematics. This sequence is defined recursively, as follows:

$$\begin{cases} a_1=1 \\ a_2=1 \\ a_n=a_{n-1}+a_{n-2} \quad n\geq 3 \end{cases}$$

In other words, the first and second terms are equal to 1 and all the rest of the terms are equal to the sum of the two preceding terms. Thus, the first terms of the sequence are: 1,1,2,3,5,8,13,21,...

Problem No. 4. Let $\{a_n\}$ be a Fibonacci sequence. Then, for any natural n, $a_n<2^n$.

Proof: The statement is true for $n=1$ and for $n=2$, since $a_1=1<2^1=2$ and $a_2=1<2^2=4$.

Assume the correctness of the statement for $n-1$ and for n, where $n\geq 2$, and prove the statement for $n+1$.

$$a_{n+1}=a_n+a_{n-1}\overset{\text{by assumption}}{>}2^n+2^{n-1}>2\cdot 2^n=2^{n+1}$$ ■

Note: In the previous problem a basis of induction containing two preceding numbers was necessary.

Problem No. 5: Let us group the uneven natural numbers in the following manner:

$\{1\}$; $\{3,5\}$; $\{7,9,11\}$; $\{13,15,17,19\}$;…, such that the n-th set contains n numbers. Prove that the sum of the numbers in the n-th set equals n^3.

Proof: The proof will be executed in two stages.

Stage 1.

We prove by induction that the first number in the n-th set, denoted $a_{n,1}$, is equal to $n(n-1)+1$.

The statement is true for $n=1$, since $a_{1,1}=1\cdot(1-1)+1=1$.

Assume the statement is true for any natural n, and prove the statement for $n+1$. In other words, we will prove that $a_{n+1,1}=(n+1)n+1$.

Each of the sets constitutes an arithmetic progression with a constant increment of 2. Therefore, the final term in the n-th set is equal to

$$a_{n,n}=a_{n,1}+2(n-1)\overset{\text{by assumption}}{=}n(n-1)+1+2(n-1)$$
$$=n^2+n-1=n(n+1)-1$$

The first number in the $n+1$-th set, is equal to the last number in the n-th set plus 2. Hence, $a_{n+1,1}=n(n+1)-1+2=(n+1)n+1$, as required.

Stage 2.

In this stage, we will prove that the sum of the terms in the n-th set is equal to n^3.

As seen in the previous stages, the n-th set constitutes an arithmetic progression containing n terms and a constant increment of 2. Thus, the sum of the terms of the n-th set is:

$$\frac{a_{n,1}+a_{n,n}}{2}\cdot n=\frac{n(n-1)+1+n(n+1)-1}{2}\cdot n=n^3 \blacksquare$$

VI. Conclusion

When teaching the subject of mathematical induction there must be emphasis on understanding the logic of the concept. The student must experiment variety of problems, so that the proof will not be too technical. It is also recommended to practice complete induction for the complete understanding of the subject.

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